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Subject: Simplified Tevatron Luminosity Equation

It has been pointed out many times that the luminosity realized in recent Tevatron operation, measured in by-now standard units of 10^{30} cm^{-2}sec^{-1} is nearly equal to the number of antiprotons used from the Recycler in units of 10^{10}. Let's see how this comes about...

Start by taking the fundamental definition of luminosity for round beams of equal intensity at energy E, and arrive at a useful expression for the Tevatron as follows:

$$\mathcal{L} = \frac{fN^2}{4\pi\sigma^{*2}} = \frac{f_0 B N^2}{4\pi(\beta^*\epsilon_N/6\pi\gamma)} = \frac{3f_0 B N^2\gamma}{2\beta^*\epsilon_N} \quad \rightarrow \quad \frac{3f_0 B\bar{N}N\gamma}{2\beta^*(\epsilon+\bar{\epsilon})/2}\cdot\mathcal{H}$$

where f_0 is the revolution frequency, B is the number of bunches in each beam, $\gamma = E/mc^2$ for the particles, β^* is the amplitude function at the interaction point, N and $\bar{N}$ are the numbers of particles per bunch of protons and antiprotons, respectively, and ϵ and $\bar{\epsilon}$ their 95% normalized transverse emittances. The "hour glass" factor, $\mathcal{H}$, is an overall reduction due to the relatively long bunches.

Now, take the result above and factor it into

$$\mathcal{L} = \frac{3f_0 B\bar{N}N\gamma}{2\beta^*(\epsilon+\bar{\epsilon})/2}\cdot\mathcal{H} \quad \longrightarrow \quad \frac{2f_0\gamma}{\beta^* r_0}\cdot\frac{3Nr_0}{2\epsilon}\cdot\frac{\mathcal{H}}{1+\bar{\epsilon}/\epsilon}\cdot B\bar{N}$$

where we have multiplied numerator and denominator by $r_0 = e^2/4\pi\epsilon_0 mc^2$, the "classical radius" of the proton.

The first factor, which has units of luminosity, is set by the geometry and optics of the ring. The second factor is the single-pass head-on beam-beam tune shift pamater, ξ. As the intensity of the proton beam is pushed upward, the proton beam emittance must be increased so as to keep the tune spread within the antiproton bunches under control. Thus, the Tevatron typically operates near the beam-beam tune shift limit of $\xi \sim 0.0125$.

The third factor contains the hour glass factor, which is a function of the ratio of β^* to the bunch length. For the present Tevatron operation $\mathcal{H}$ is about 0.6. Changing this requires at least a change in the RF frequency (higher) at collision and is not foreseen for the Tevatron operation at this time. The denominator of this factor contains the ratio of antiproton to proton emittances, which one tries to optimize to keep the tune spreads of the two beams in line while under the influence of the beam-beam interactions. Typically, this ratio is about 4/7. The fourth and final factor is, of course, the total number of antiprotons in the ring, which is some fraction (~85%) of those taken from the Recycler to set up the store.

So, we put in the typical numbers:

$$\begin{aligned}\mathcal{L} &= \frac{2\cdot 47.7\times 10^3/\mathrm{s}\cdot 1045}{1.5\times 10^{-16}\ \mathrm{cm}\cdot 30\ \mathrm{cm}} \quad\cdot\quad \frac{3/5}{1+4/7} \quad\cdot\quad \frac{1}{80}\cdot\left(\frac{B\bar{N}}{10^{10}}\right)\cdot 10^{10} \\ &\approx \left(10^{30}\mathrm{cm}^{-2}\mathrm{s}^{-1}\right)\ \cdot\ \left(\frac{B\bar{N}}{10^{10}}\right).\end{aligned}$$

If we start with an antiproton "stash" of 375×10^{10} in the Recycler, and 85% make it to low-beta and collisions, then the initial luminosity should be about $320\times 10^{30}/\mathrm{cm}^2/\mathrm{s}$, consistent with operational experience.